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III. GRAVITATIONAL FIELDS

Determination of the Earth's gravitational field from satellite orbits: methods and results

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The structure of theories used in determining the gravitational field from the perturbations of orbits of artificial satellites is discussed and it is shown how it corresponds to the fact that small departures from a Keplerian ellipse are readily observed. Some current problems are mentioned. Statistical problems in the estimation of parameters of the field from orbital data are considered and recent estimates are summarized.

1. Introduction

It will not be necessary to recount how observations of the very first artificial satellites greatly improved our knowledge of the external gravity field of the Earth, nor how, since that time, this knowledge has rapidly become more refined and detailed, for that is well known. The important geodetic and geophysical applications of the new data are also common knowledge.

The subjects of this review are the theoretical methods available for the determination of the potential from observations of orbits, the numerical and statistical problems that arise in applying those methods to available observations, and an indication of the range and reliability of the results we now have, together with some remarks on the geophysical implications they bear.

It is important to emphasize a restriction which can be imposed on the theories. We shall concentrate attention on those features of an orbit which depend on the small departures of the external potential from that of a point mass and we wish, if possible, to isolate these features from those dependent on atmospheric drag or other factors in which, for the present purpose, we are not interested. The theory required for the study of the potential is thus not necessarily a complete theory of orbits in the sense that it would enable the position of a satellite at any given time to be calculated from the initial conditions; the more restricted aim will be to classify and calculate the departures from a simple orbit. The potential of the Earth is dominated by the term of zero order, $-\mu/r$ ($\mu = fM$, where f is the gravitational constant and M the mass of the Earth), but the greatest interest, geophysically and geodetically, attaches to the small departures from this term; orbits in the potential $-\mu/r$ are ellipses, so that the principal purpose of the theory will be to study the departures from ellipses that arise from the small terms of higher order in the potential.

Expressed as a series of spherical harmonics, the potential is usually written as

$$-\frac{\mu}{r}\left[1-\sum_n \left(\frac{a_e}{r}\right)^n \{J_n P_n(\cos\theta)+\sum_m (C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)P_n^m(\cos G)\}\right],$$

where (r, θ, λ) are spherical polar coordinates and a_e is the equatorial radius of the Earth.

2. Structure of orbital theories

The Hamiltonian formulation of dynamics is particularly well suited for studying the effects of higher order terms in the potential. The equations of motion in canonical form read

$$\dot{p}_r = \frac{\partial F}{\partial q_r}, \quad \dot{q}_r = -\frac{\partial F}{\partial p_r},$$

where, in the usual notation, p_r, q_r, are conjugate momentum and coordinate variables and F is the Hamiltonian, equal to the kinetic energy T plus the potential energy V.

It may happen that the Hamiltonian is independent of one or more of the momenta or coordinates; if p_r, for example, does not appear in F, then $\dot{q}_r$ is a constant of the motion. For this to happen the potential must be of a particularly simple form, V_0 say, the corresponding Hamiltonian being F_0. A potential, V, which does not differ much from V_0, may be written as V_0+V_1 and if the Hamiltonian is F_0+F_1, the equations of motion in V are

$$\dot{p}_r = \frac{\partial(F_0+F_1)}{\partial q_r}, \quad \dot{q}_r = -\frac{\partial(F_0+F_1)}{\partial p_r}.$$

But since $\partial F_0/\partial p_r = 0$,

$$\dot{q}_r = -\partial F_1/\partial p_r$$

and so

$$q_r = \int^t \frac{\partial F_1}{\partial p_r} \mathrm{d}t + \text{a constant.}$$

The point of this formulation is that if V_1 is small, it may be easy to calculate the integral.

However, more significant is the fact that this formulation of the equations of motion concentrates attention on the matters of interest, namely the departures of an orbit from a known orbit in an elementary potential V_0. $-\mu/r$ is such an elementary potential and the orbits in it are ellipses which are of constant form and orientation in space; the orbital elements a (semi-major axis), e (eccentricity), ω (longitude of perigee), i (inclination) and Ω (longitude of node) are constants, while the mean anomaly l is equal to nt and increases steadily with time. Thus it must be possible to write the Hamiltonian for this dynamical system in such a way that it is a function of one parameter only, namely that conjugate to the mean anomaly. The elliptic elements are not themselves a set of canonically conjugate variables but the pairs in the following set, the Delaunay variables, are canonically conjugate:

$$L = (\mu a)^{\frac{1}{2}}, \qquad l = nt;$$

$$G = L(1-e^2)^{\frac{1}{2}}, \qquad g = \omega;$$

$$H = G\cos i, \qquad h = \Omega.$$

The momentum constants, L, G, H, are the integral constants of the motion, namely, the energy, the total angular momentum and the angular momentum about a prescribed axis (that for which $i = \frac{1}{2}\pi$).

Corresponding to the potential $-\mu/r$, the position of a satellite would be given in spherical polar coordinates (r, θ, λ), with corresponding momenta $(p_r, p_\theta, p_\lambda)$. Although the potential is independent of θ and λ, the kinetic energy is dependent on the momenta; thus none of the coordinates is a constant of the motion and this is not a suitable system for representing small departures from an elliptic orbit. The transformation to the Delaunay variables can be derived by a general method. Suppose that a dynamical system can be described by two sets of canonical momenta and coordinates (p_r, q_r) and (P_s, Q_s), say. They will then be related by the expression

$$\Sigma(P_s \mathrm{d}Q_s - p_r \mathrm{d}q_r) = \mathrm{d}S,$$

where S is the transforming function, a function of (p_r, q_r) and (P_s, Q_s) in general.

Then if p_r, q_r and S are known, $P_s = \partial S/\partial Q_s$.

The Hamilton–Jacobi partial differential equation enables a transforming function S to be found such that all but one of the momenta and coordinates P_s, Q_s, are constants, the remaining one being a linear function of time. This equation simply states that if F is a constant, α_n, say, then all the rates of change of the transformed variables, except one, will vanish. The equation reads

$$F\left(q_1 \dots q_n, \frac{\partial S}{\partial q_1} \dots \frac{\partial S}{\partial q_n}\right) = \alpha_n.$$

If a solution exists, it will involve n arbitrary constants α_r which may be equated to the Q_r, while the momentum constants, $\beta_r = -\partial S/\partial \alpha_r$, except for $\beta_n = t - \partial S/\partial \alpha_n$.

The Hamilton–Jacobi equation admits of solutions only in special cases. The coordinate system must be one of the eleven in which Laplace's equation separates and the potential must be of proper form. For spherical polar coordinates, the potential is $-\mu/r$ and the transformed momenta and coordinates are the Delaunay variables.

Some time has been spent on this discussion of the Hamiltonian equations because it shows that the changes in the orbital elements, or more strictly in the Delaunay variables, are naturally and directly related to the departure of the actual potential from the form $-\mu/r$.

The equations of motion for a potential V equal to $-\mu/r + V_1$ are then

$$\dot{L} = \partial V_1/\partial l, \quad \dot{l} = -\partial V_1/\partial L + n,$$

and so on (n is the mean motion of the satellite).

To solve them, V_1 must be expressed as a function of the Delaunay variables. That is rather inconvenient. It is natural to write the potential outside the Earth as a series of spherical harmonics, functions of (r, θ, λ) and when written as functions of the Delaunay variables, the expressions are cumbersome. Somewhat simpler expressions arise when the harmonics are written in terms of the elliptic elements and it is therefore usual for first-order theories to transform the Hamiltonian equations of motion for the Delaunay variables to equations of motion for the elliptic elements.

The transformations are elementary and lead to the Lagrangian equations:

$$\dot{a} = \frac{2}{na}\frac{\partial V_1}{\partial l}, \qquad \dot{e} = \frac{2}{na^2 e}\frac{\partial V_1}{\partial l} - \frac{\eta}{na^2 e}\frac{\partial V_1}{\partial \omega},$$

$$\dot{\omega} = \frac{\eta}{na^2 e}\frac{\partial V_1}{\partial e} - \frac{\cot i}{na^2 \eta}\frac{\partial V_1}{\partial i}, \qquad \dot{\Omega} = \frac{1}{na^2 \eta \sin i}\frac{\partial V_1}{\partial i},$$

$$\partial i/\partial t = \frac{\cot i}{na^2 \eta}\frac{\partial V_1}{\partial \omega} - \frac{1}{na^2 \eta \sin i}\frac{\partial V_1}{\partial \Omega}, \qquad \dot{l} = n - \frac{\eta^2}{na^2 e}\frac{\partial V_1}{\partial e} - \frac{2}{na}\frac{\partial V_1}{\partial a},$$

$$(\eta^2 = 1 - e^2).$$

Except for the term proportional to the second zonal harmonic P_2 (cos θ), all harmonic terms in the potential are of order 10^{-6} or less of the zero-order term, $-\mu/r$; a first-order theory is therefore adequate for them all and the only difficulties are the algebraic ones of writing a term such as P_{20} (cos θ) as a function of cos i. The second zonal harmonic is of order 10^{-3} of $-\mu/r$, its square is of the order of the higher harmonics, and its effects must therefore be calculated with a second-order theory in which the elements that enter the right sides of the equations of motion are not assumed to be constant. Such theories have been given for the Lagrangian equations (Merson 1961; Kozai 1959; Zhongholovitch & Pellinen 1962).

In first-order theories, exact definitions of the orbital elements are not required but they must be considered in second-order theories. The difficulties arise from the fact that the elements change with the position of the satellite in the orbit. The elliptical elements are osculating elements, that is they are the elements of that elliptic orbit, the arc of which coincides in position and velocity with the arc of the actual orbit at the position of the satellite; if the satellite is not following an elliptic orbit the osculating elements change as the satellite goes round the orbit. It is convenient to remove some of the short-term variations by dealing with elements averaged over the orbit and then the exact way in which the average is taken affects the form of the results of a second-order theory. The theories mentioned thus give results which superficially are somewhat different but it has been shown that they are in fact equivalent (Cook 1963).

The algebraic complexity of the second-order theories is considerable and the Lagrangian equations are no longer clearly superior to the Hamiltonian equations, particularly since the latter admit a treatment which helps to clarify the definition of the variables in a second-order theory. This treatment, first applied to artificial satellite theory by Brouwer (1959), is due to von Zeipel and is an extension of the transformation theory already used to find variables of which the Hamiltonian is independent.

If the potential is taken to be just $-\mu/r[1+(a_e/r)^2 J_2 P_2 (\cos\theta)]$, where a_e is the equatorial radius of the Earth, the Hamiltonian is a function of the energy and angular momenta, L, G, H, of the mean anomaly and of the longitude of the node, h. The aim of von Zeipel's transformation is to find transformed variables, denoted by primes, L' ..., such that the Hamiltonian is independent of say l' as well as of h'. The procedure is to derive the new variable from the old by a transforming function S which is assumed to be close to the identical transformation, S_0; thus

$$S = S_0+S_1,$$

where S_1 is small, of order J_2.

The transformed Hamiltonian F^* differs from the original form F only through the difference between L and L' and so by Taylor's theorem,

$$F^* = F+(L'-L)\,\partial F/\partial L.$$

V may be taken to be $-\mu/r$ in calculating $\partial F/\partial L$, while

$$L' = \partial S/\partial l' \quad \text{and} \quad L = \partial S_0/\partial l,$$

so that $L'-L = \partial S_1/\partial l$ to terms of first order in J_2.

Thus
$$\frac{\partial S_1}{\partial l} = \frac{F^*-F}{\partial F/\partial L}.$$

$\partial F/\partial L$ is $-\mu^2/L^3$ and F^*-F is $-(\mu/r)\,J_2\,(a_e/r)^2 P_2(\cos\theta)$, which can be written in terms of the Delaunay variables.

Hence S_1 can be found by a single direct integration with respect to l; all other transformed variables can then be found by differentiation:

$$G = G' + \partial S_1/\partial g, \quad \text{for example.}$$

A further transformation leads to a Hamiltonian F^{**} that is independent of g' also, that is, it is a function of the transformed momenta, L'', G'', H'', alone. But these are constants because

$$\frac{\partial H''}{\partial t} = -\frac{\partial F^{**}}{\partial g''} = 0, \quad \text{for example.}$$

The rates of change of the transformed coordinates then follow directly

$$\dot{g}'' = \partial F^{**}/\partial G'' \quad \text{and so on,}$$

and from these, by retracing the transformation, the behaviour of the original osculating variables, l, g, h, may be recovered.

The advantage of this way of handling the equations of motion is that the terms that are periodic in l (short period terms) and g (long period terms) are handled systematically, and the definitions of the various elements are clear.

The transformed coordinates, such as g'', h'', are comparable with the mean osculating elements used in Lagrangian theories, though not always identical. Thus, Brouwer's results are formally different from those of Lagrangian theories but here again it has been shown that they are equivalent.

An effectively complete first-order theory of perturbation of an elliptic orbit is available for zonal harmonics, that is, the procedure for calculating secular, long-period and short-period terms corresponding to any zonal harmonic in the potential is known and explicit results are available for harmonics up to about the 22nd order. In addition, the terms proportional to J_2^2 have been derived in a number of different ways. The perturbations fall into distinct classes according to whether they arise from even zonal harmonics, odd zonal harmonics or tesseral harmonics. The dominant terms arising from the even zonal harmonics are secular changes of the longitude of node and perigee; all elements contain terms of argument 2ω, which are however of order e^2. The principal terms arising from the odd zonal harmonics are long periodic terms, of argument ω, and order e in node, perigee, inclination, eccentricity and perigee distance. Short-period terms are present in all perturbations but tesseral harmonics give rise to neither secular nor long-period terms; the principal perturbations due to tesseral harmonics have arguments that are multiples of $(\phi-\Omega)$, where ϕ is equal to the longitude of an observing site plus sidereal time.

The theory of orbits has thrown up a number of intricate problems, in particular, that of the behaviour of a satellite in an orbit with the critical inclination, given by

$$\cos^2 i = \tfrac{1}{5},$$

for which the rate of rotation of perigee is zero. While of great theoretical interest, this problem, which has not been fully clarified, does not seem to arise in practice.

The problem of commensurable orbits, another situation in which the simple first-order theory fails, is now, on the other hand, of great practical importance, for it has become possible to determine the term involving $P_2^2(\cos\theta)$ as well as terms of quite high order from observations of commensurable orbits. In these orbits, the ratio of the mean motion of the satellite, n, to the speed of rotation of the Earth, $\tilde{\omega}$, is a rational number. If the potential includes tesseral harmonics, it varies periodically with time at any point fixed in an inertial coordinate system and the Hamiltonian is an explicit function of time. If the Delaunay variables are denoted by the suffix D, new variables may be chosen appropriate to coordinates rotating with the Earth, namely

$$l = l_D, \quad g = g_D, \quad h = h_D - \tilde{\omega}(t-t_0).$$

The conjugate momenta must satisfy the condition that

$$L\,\mathrm{d}l + G\,\mathrm{d}g + H\,\mathrm{d}l - L_D\,\mathrm{d}l_D - G_D\,\mathrm{d}g_D - H_D\,\mathrm{d}h_D - (F_D - F)\,\mathrm{d}t$$

should be a perfect differential. The necessary relations are

$$L = L_D, \quad G = G_D,$$
$$H = H_D, \quad F = F_D + \tilde{\omega}H.$$

If V_{pq} is a potential term proportional to $P_p^q(\cos\theta)\cos q(\lambda-\beta_{pq})$, β_{pq} being a phase angle,

$$F = (\mu^2/2L^2) + V_{pq} + \tilde{\omega}H.$$

Let von Zeipel's transformation be applied to determine the orbital variations due to the term V_{pq}; V_{pq} can be expanded as a series of terms like

$$\alpha k_1 k_2 \cos[k_1 l + k_2 g + q(\lambda - \beta_{pq})].$$

The transforming function may similarly be written as the identical-transformation plus a series of terms like

$$\gamma k_1 k_2 \cos[k_1 l + k_2 g + q(\lambda - \beta_{pq})],$$

and on applying von Zeipel's transformation it will be found that

$$\gamma k_1 k_2 = \frac{\alpha k_1 k_2}{k_1(\mu^2/L^{*3}) - q\tilde{\omega}} \quad \text{(Morando 1962).}$$

If $\tilde{\omega}$ and μ^2/L^{*3}, which is equal to the mean motion, are commensurable, there will always be one term for which the denominator vanishes and the transformation cannot be applied.

In that case, secular perturbations arise, or more generally, if the orbit is nearly commensurable, the perturbations are periodic, the periods being longer the closer the motion is to being commensurable.

The following table (Yionoulis 1965) shows the orbital period and altitude at which commensurability occurs for a given degree of tesseral harmonic:

m	period (min)	altitude (km)
2	718	20200
5	287·2	6180
8	179·5	4160
9	159·6	3360
10	143·6	2710
11	130·0	2120
12	119·7	1670
13	110·5	1240
14	102·6	880

In the last few years, the theory of commensurable orbits has been considerably developed, especially in its application to synchronous orbits and geostationary satellites. (See Allan 1965; Anderle 1965; Yionoulis 1965, 1966*a*, *b*.)

It is something of a disadvantage that the second zonal harmonic, which is about 10^3 times greater than any other, should have to be treated by an approximate method; Vinti (1959) has shown how it may be incorporated in an exact treatment. The Hamilton–Jacobi equation admits of an exact solution by separation of variables in oblate spheroidal coordinates provided the potential (in spherical coordinates) is of the form

$$-\mu/r\,[1-(a_e/r)^2 J_2 P_2\,(\cos\theta)+\ldots+(-)^{2n}\,(a_e/r)^{2n}\,(J_2)^{2n} P_{2n}\,(\cos\theta)+\ldots],$$

There are two disposable constants in this form, μ and J_2, which may be chosen to fit the actual values for the Earth. It then happens that the coefficient of P_4 is close to the actual value but all other actual coefficients are much greater than in the Vinti potential. Thus the dominant second zonal harmonic can be treated exactly but perturbation methods must still be used for all others.

The orbit may then be described by variables similar to the Delaunay variables, and if the potential is of the Vinti form all but one of those variables will be constant, the remaining one being a linear function of time.

It should then be possible to compare the results of Vinti's theory for the P_2 term with that obtained from the second-order perturbation theory described earlier. The connexions between the canonical variables in Vinti's theory and the elliptic elements are however rather intricate and, although Iszak (1962) had explored some of them, the work has not been taken far enough to show explicitly whether Vinti's theory and second-order perturbation theory do in fact give the equivalent results for, say, the behaviour of the node.

It is not possible to ignore all other forces and obtain good estimates of the Earth's potential. To first order, the resistance of the air does not affect the motion of node and perigee, but there are second-order interactions together with larger effects on eccentricity and inclination, so that an adequate theory of the effects of air resistance is required to enable corrections to be applied. This is important for close satellites. Radiation pressure and the attraction of the Sun and the Moon are important at greater distances. First-order perturbation theory is adequate but the algebra is quite complex and cannot always be carried out literally for radiation pressure.

3. Estimation problems

Analysis of the motion of an artificial satellite gives an amplitude for the term of a particular periodicity in the perturbation of some element that may be compared with that calculated from the theories outlined in the previous section. Thus, to take the secular motion of the node, the observed change in one nodal period, $\delta\Omega_s$, say, may be written as

$$\delta\Omega_s = a_2 J_2 + a_4 J_4 + \ldots + a_{2n} J_{2n} + \ldots + \alpha J_2^2 + \delta\Omega_{LS} + \delta\Omega_R + \delta\Omega_A,$$

where $\delta\Omega_{LS}$ is the luni-solar part due to the attraction of the Sun and the Moon, $\delta\Omega_R$ is the part due to solar radiation pressure and $\delta\Omega_A$ is the part due to atmospheric interaction.

These three terms can all be calculated, as can αJ_2^2, since J_2 is well enough known for the term to be estimated with sufficient accuracy.

Each satellite can therefore provide an observation equation

$$a_{j2} J_2 + a_{j4} J_4 + \ldots + a_{j2n} J_{2n} = y_j.$$

The parameters $a_{j2} \ldots a_{j2n}, \ldots$ are functions of the orbital elements.

The major problem in estimating the coefficients J_{2n} from such sets of observation equations is that there are more coefficients to be found than there are distinct observation equations.

Most of the satellites that have been launched are unsuitable for determining the potential because they have such short lives that the orbital elements cannot be found with sufficient accuracy. Of the usable orbits, few are clearly distinct. If a set of observation equations leads to a singular set of normal equations for the parameters to be determined, the addition of a further observation equation that is a multiple of an existing one will not reduce the singularity of the normal equations. The addition of an equation which is nearly a multiple of an existing one may formally remove the singularity but will leave the normal equation ill-conditioned, so that the estimates of the parameters will be strongly correlated and have large uncertainties. The best determination will come from observation equations in which the coefficient of a different parameter dominates each separate equation while the worst determinations will come from equations in which the relative magnitudes of the coefficients are similar in all equations.

The factors a_n in the observation equations are functions of the elements a, e and i. a enters in the factor $(a_e/a)^n$ and for close satellites with $a \sim a_e$, does not vary greatly up to $n \sim 12$. However, if a exceeds about 8000 km, harmonics beyond the 6th can be ignored. e enters as some power of $(1-e^2)$ (for the secular terms) and as e is usually quite small, variations in e have little influence on the a_n. The dominant influence is that of i and the greatest differences between observation equations are obtained by taking orbits with many different inclinations distributed as uniformly as possible from 0 to 90°. However, most orbits fall into some seven rather narrowly limited bands and so there are in effect only some seven clearly distinct observation equations for close satellites. It is now clear that this is less than the number of even or odd zonal harmoc terms that must be retained to give an adequate description of the potential. A study of the secular motions of node and perigee and of the estimates of the even zonal harmonics (Cook 1965) showed that harmonics beyond the 12th could not be ignored. By using orbits of semi-major axes large enough for the products $a_8 J_8$, $a_{10} J_{10}$, ... to be ignored, the coefficients J_2, J_4 and J_6 can be found with quite small errors. Estimates of J_8, J_{10}, J_{12} and J_{14}, made from data on other satellites, are available but must be considered to have large uncertainties and to be strongly correlated; the correlation is due to the fact that each harmonic occurs significantly in all observation equations and that the number of observation equations exceeds by only 1 or 2 the number of harmonics being estimated.

The conclusions apply to all three classes of harmonic. There is an infinite set of harmonics that give rise to perturbations of given periodicity, secular, long term or short term of argument $q(\phi-\Omega)$. There is only a limited number of satellites of distinct inclination and unless satellites with a wide range of semi-major axis can be used, there will be more significant products $a_n J_n$ ... than there are observation equations. D. G. King-Hele, G. E. Cook & D. W. Scott (this volume, p. 144) have been able to estimate a greatly extended set of odd harmonics by making use of orbits with a wide range of semi-major axes as well as of inclination, but generally it must still be considered that we know little more than the orders of magnitude of most harmonic coefficients.

In view of the difficulties of estimating the coefficients of spherical harmonics, it is sometimes suggested that an alternative representation be used. It is possible to make a minor change by using the algebraic direction cosine form of spherical harmonics, for which a close satellite theory similar to the Hill–Brown lunar theory was given by Brouwer. Any set of functions other than spherical harmonics is inconvenient because they are not orthogonal. The main problem would not, however, be affected by any such re-formulation: the difficulty lies in the number of independent parameters needed to represent the potential adequately. A satellite at a height of say 500 km above the surface will experience an attraction corresponding to the average surface gravity field over an area of approximately 300×300 km, or about $15^\circ \times 15^\circ$. The surface of the Earth is covered by some 400 such areas and it is known from statistical studies of surface gravity that while the mean values in areas of such dimensions are to some extent correlated, corresponding to the existence of harmonics of low order in the potential, there is a substantial uncorrelated component. It follows that, however the potential is represented, well over 100 parameters are needed.

4. Numerical values

(a) *Even zonal harmonics*

Observation equations for the even zonal harmonic coefficients are the easiest to obtain because they use the secular changes which can be obtained with great accuracy by simple methods of observation extended over a long time.

The term proportional to J_2^2 must be properly calculated: this means that careful attention has to be given to the definition of orbital elements and the relation of them to the scheme for the reduction of observations.

Recent results are summarized in table 1. The estimates by Cook (1965) are obtained from orbits with semi-major axes large enough for the influence of the eighth and higher harmonics to be small. It was concluded in the same study that actual values of these higher harmonics could not be estimated from the available data but that the order of magnitude of the coefficients of these harmonics was about $0{\cdot}2 \times 10^{-6}$. The range of estimates in table 1 appears to confirm this conclusion.

Table 1. Values of even zonal harmonics

coefficient	A. H. Cook (1965)	King-Hele & Cook (1965)	Kozai (1964)	Smith (1965)
10^6 J_2	1082·65	1082·68	1082·64	1082·64
J_4	−1·60	−1·61	−1·65	−1·70
J_6	+0·73	+0·71	+0·65	+0·73
J_8	—	+0·13	−0·27	−0·46
J_{10}	—	+0·09	−0·05	−0·17
J_{12}	—	−0·31	−0·36	−0·22
J_{14}	—	—	+0·18	+0·19

J_n is the coefficient of $(a_e/r)^n P_n(\cos\theta)$.

(b) *Odd zonal harmonics*

Estimates of the odd zonal harmonics are given in table 2. The results reported to this meeting by D. G. King-Hele and his collaborators have greatly extended our knowledge of these harmonics.

TABLE 2. VALUES OF ODD ZONAL HARMONICS

coefficient	Smith (1963)	Kozai (1964)	Guier & Newton (1965)	King-Hele, Cook & Scott (1967)
10^6 J_3	−2·44	−2·55	−2·69	−2·50
J_5	−0·18	−0·21	−0·01	−0·26
J_7	−0·30	−0·33	−0·63	−0·40
J_9	—	−0·45	+0·21	0·0
J_{11}	—	+0·30	—	−0·27
J_{13}	—	−0·11	—	+0·36
J_{15}	—	—	—	−0·65
J_{17}	—	—	—	+0·30
J_{19}	—	—	—	0·0
J_{21}	—	—	—	+0·58

J_n is the coefficient of $(a_e/r)^n\, P_n(\cos\theta)$.

(c) *Tesseral harmonics*

Tesseral harmonics are particularly difficult to estimate. Short period perturbations are more difficult to observe than long period or secular perturbations and they involve knowing the geocentric position of the observing station. Thus the tesseral harmonics cannot be found independently of the station positions and the results of Iszak, for example, are obtained from an adjustment in which tesseral harmonics and corrections to station coordinates are determined simultaneously.

Some determinations are available from commensurable orbits. The $P_2^2(\cos\theta)$ term has been estimated from the librating position of a geostationary satellite (Allan 1964).

Commensurability effects have been found (Anderle 1965; Yionoulis 1965, 1966*a*, *b*) in the orbits of the following satellites:

	inclination (deg)	nodal period (min)	m
1961 o1	67	103·8	14
1962 $\beta\mu$1	50	107·8	13
1963 49B	90	107·2	13
1964 26A	90	103·1	14

The periods of the commensurability terms range from 2·5 to 14 days. Effects of higher order commensurability with $m = 27$ have been observed (Anderle 1965).

The results of the analyses of these orbits are included in table 3. They are made possible because of the high accuracy of along-track and slant range Doppler tracking data.

Generally speaking, the scatter of estimates of the tesseral harmonics is very great, once again owing to the fact that the harmonics which have not been determined are not negligible.

(d) *Normalization, orders of magnitude*

Many different conventions for choosing the numerical coefficients of spherical harmonics will be found in the literature. Following general practice, the zonal harmonics are expressed in terms of the Legendre coefficients, $P_n(\cos\theta)$,

for which
$$\int_0^\pi \{P_n(\cos\theta)\}^2\,\mathrm{d}(\cos\theta) = \frac{2}{2n+1},$$

and so

$$\int_S \{P_n(\cos\theta)\}^2\,\mathrm{d}S = \frac{4\pi}{2n+1},$$

where the integral is taken over the surface of the unit sphere.

Table 3. Values of tesseral harmonics (coefficients $\times 10^6$)

		Iszak (1966)		Guier & Newton (1965)		Anderle (1965)		Yionoulis (1965, 1966 *a*, *b*)	
n	m	$\bar{C}_{nm}$	$\bar{S}_{nm}$	$\bar{C}_{nm}$	$\bar{S}_{nm}$	$\bar{C}_{nm}$	$\bar{S}_{nm}$	$\bar{C}_{nm}$	$\bar{S}_{nm}$
2	2	2·08	−1·25	2·38	−1·20	2·45	−1·52	—	—
3	1	1·60	−0·04	1·84	0·21	2·15	0·27	—	—
	2	0·38	−0·08	1·22	−0·68	0·98	−0·91	—	—
	3	−0·17	1·40	0·66	0·98	0·58	1·62	—	—
4	1	−0·38	−0·40	−0·56	−0·44	−0·49	−0·57	—	—
	2	0·20	0·58	0·42	0·44	0·27	0·67	—	—
	3	0·69	−0·10	0·84	0·00	1·03	−0·25	—	—
	4	−0·11	0·43	−0·21	0·19	−0·41	0·34	—	—
5	1	−0·14	−0·04	0·14	−0·17	0·03	−0·12	—	—
	2	0·24	−0·27	0·27	−0·34	0·64	−0·33	—	—
	3	−0·67	0·05	0·09	0·10	−0·39	−0·12	—	—
	4	−0·13	0·16	−0·49	−0·26	−0·55	0·15	—	—
	5	0·08	−0·41	−0·03	−0·67	0·21	−0·59	—	—
6	1	−0·02	0·12	0·00	0·10	−0·08	0·19	—	—
	2	0·05	−0·23	−0·16	−0·16	0·13	−0·46	—	—
	3	0·05	0·00	0·53	0·05	−0·02	−0·13	—	—
	4	0·07	−0·39	−0·31	−0·51	−0·19	−0·32	—	—
	5	−0·28	−0·38	−0·18	−0·50	−0·09	−0·79	—	—
	6	−0·12	−0·59	0·01	−0·23	−0·32	−0·36	—	—
7	1	—	—	0·13	0·09	0·33	0·08	—	—
	2	—	—	0·46	0·06	−0·35	−0·19	—	—
	3	—	—	0·39	−0·21	0·32	0·04	—	—
	4	—	—	−0·14	0·00	−0·47	−0·24	—	—
	5	—	—	−0·06	−0·19	0·05	0·02	—	—
	6	—	—	−0·45	−0·75	−0·48	−0·24	—	—
	7	—	—	0·09	−0·14	—	—	—	—
8	1	—	—	−0·15	−0·05	—	—	—	—
	2	—	—	0·09	−0·04	—	—	—	—
	3	—	—	−0·05	0·22	—	—	—	—
	4	—	—	−0·07	−0·04	—	—	—	—
	5	—	—	0·08	−0·00	—	—	—	—
	6	—	—	−0·02	0·67	—	—	—	—
	7	—	—	0·17	−0·07	—	—	—	—
	8	—	—	−0·15	0·07	—	—	—	—
13	12	—	—	—	—	—	—	0·09	0·01
	13	—	—	−0·046	0·048	−0·03	0·11	0·10	0·01
14	14	—	—	—	—	—	—	0·07	0·02
15	13	—	—	—	—	−0·06	−0·06	0·05	0·02
	14	—	—	—	—	0·01	−0·03	0·01	0·00

$\bar{C}_{nm}$, $\bar{S}_{nm}$ are coefficients of the harmonics $\bar{Y}_{nm} = \bar{P}_{nm}(\cos\theta)(\cos m\lambda, \sin m\lambda)$ which are normalized such that

$$\int_{\text{unit sphere}} \bar{Y}^2_{nm}\,\mathrm{d}s = 4\pi.$$

The simplest convention for the tesseral harmonics is that suggested by Kaula; in it the integral of the square of a surface harmonic over a unit sphere is 4π; that is, if $Y_{nm}(\theta, \lambda)$ is a surface harmonic in this convention

$$\int_{\text{sphere}} Y^2_{nm}\,\mathrm{d}S = 4\pi.$$

If $\overline{C}_{nm}$, $\overline{S}_{nm}$, are the coefficients of cosine and sine terms in this convention and if C_{nm}, S_{nm} are the coefficients of $P_n^m(\cos\theta)$ [$\cos m\lambda$, $\sin m\lambda$] then

$$\begin{pmatrix}\overline{C}^2_{nm}\\ \overline{S}^2_{nm}\end{pmatrix} = \frac{1}{2(2n+1)}\frac{(n+m)!}{(n-m)!}\begin{pmatrix}C^2_{nm}\\ S^2_{nm}\end{pmatrix}.$$

TABLE 4. SUMS OF SQUARES OF AMPLITUDES OF HARMONIC COEFFICIENTS

n	$\sum_m \overline{C}^2_{nm}$	$(\sum_m \overline{C}^2_{nm})^{\frac{1}{2}}$
	10^{-12}	10^{-6}
2	10·7	3·3
3	8·0	2·8
4	2·3	1·5
5	0·9	0·95
6	0·7	0·84
7	0·8	0·9
8	0·6	0·77
13 to 20		about 0·2

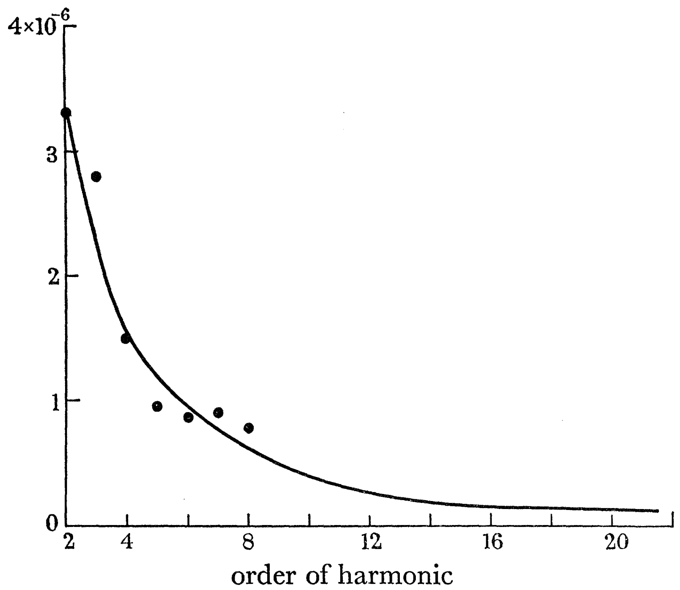

FIGURE 1. Root sum of squares of harmonic amplitudes as function of order. Normalization:

$$\int_{\text{unit sphere}} \overline{Y}^2_{nm}\,\mathrm{d}S = 4\pi.$$

The numerical advantage of this convention is that the sum square amplitude of all harmonics of order n is simply

$$\sum_m (\overline{C}^2_{nm}+\overline{S}^2_{nm}).$$

In table 4 an attempt has been made to estimate these total mean square amplitudes and an idea of the behaviour is sketched in figure 1.

5. Conclusion

While the theory of satellite orbits is well developed and adequate for the determination of tesseral as well as zonal harmonics, the available data are sufficient to give reliable estimates of only a few harmonics, although fairly definite ideas of the orders of magnitude of other harmonics are becoming clear. Much better estimates of the zonal harmonics would be possible if orbits with inclinations less than 28° were available. The work of D. G. King-Hele, G. E. Cook & D. W. Scott (this volume, p. 144) and that of Anderle (1965) on high-order tesseral harmonics both show that great advances in our knowledge of the harmonic coefficients are still possible and may be expected.

Although some detailed estimates of the harmonics may be unreliable for use in geophysical studies, much can be learnt from the orders of magnitude and from correlations of the harmonics of low order with other physical properties of the Earth. It is clear, and recent results greatly strengthen the conclusion, that there must be appreciable density anomalies in the mantle. In particular the harmonics beyond the 12th or 13th order cannot arise from irregularities of the core-mantle boundary. On the other hand, a lack of correlation with the distribution of continents and oceans shows that the sources must lie below the crust and probably below the uppermost part of the mantle but a correlation with heat flow combined with a consideration of the maximum depth at which the heat sources can lie suggests that the sources of both the irregular gravity field and of the heat flow lie in the upper part of the mantle. These are certainly valuable inferences.

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PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Tesseral harmonics of the Earth's gravitational field from camera tracking of satellites

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[Abstract]

A total of 7234 Baker-Nunn camera observations of five satellites were analysed to determine simultaneously 44 tesseral harmonic coefficients of the gravitational field, 36 station coordinates, and 511 orbital elements. Supplementary observational data incorporated in the solution included accelerations of 24 h satellites and directions between tracking stations from simultaneous observations; observation equations were also written for the differences between geometrical and gravitational geoid heights at tracking stations. Several variations in relative weighting of different observational data and *a priori* variances of parameters were tested. The previous independent solution most closely approached was that by Anderle based on Doppler data, from which the r.m.s. discrepancy was $\pm 0{\cdot}18 \times 10^{-6}$ for 38 normalized harmonic coefficients, or ± 7 m in total geoid height. An equatorial radius of 6378160 ± 5 m was obtained.

The complete paper is published in *J. Geophys. Res.* **71**, 4378 (1966).

Tests and combination of satellite determinations of the gravity field with gravimetry

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[Abstract]

Six solutions for the tesseral harmonic coefficients of the geopotential from satellite orbit analysis were compared to terrestrial gravimetry in the form of mean free-air anomalies of 300 n. mi. squares. Statistical parameters calculated were the mean square values of each type of determination as well as the mean square difference for different samples based on the number of observations in the 300 n. mi. square.

The study showed significant variation in quality between different solutions, the best being that recently obtained from camera tracking of satellites by M. Gaposhkin of the Smithsonian Astrophysical Observatory. It also showed that the arithmetic mean of four independent satellite solutions was better than any single solution. This combined satellite solution was used in an adjustment with the gravimetry to obtain a single best solution, in the forms of a geoid map; spherical harmonic coefficients to degree and order 12, 12; and gravity anomalies at 10° intervals.

The complete paper is published in *J. Geophys. Res.* **71**, 5303 (1966).

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